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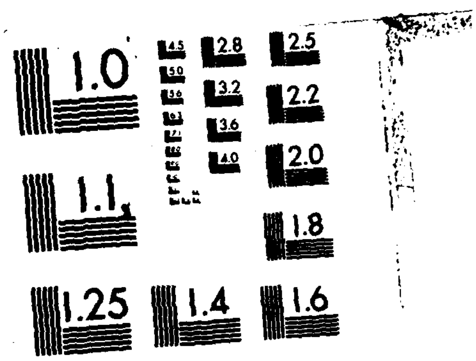

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PLASTIC DEFORMATIONS NEAR A RAPIDLY PROPAGATING CRACK TIP

by

J. D. Achenbach and Z. L. Li

Department of Civil Engineering
Northwestern University
Evanston, IL. 60201

Office of Naval Research

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ABSTRACT

For rapid crack propagation in an elastic perfectly-plastic material, explicit expressions have been obtained for the dynamic strains on the crack line, from the moving crack tip to the moving elastic-plastic boundary. The method of solution uses power series in the distance to the crack line, with coefficients which depend on the distance to the crack tip. Substitution of the expansions in the equations of motion, the yield condition (Huber-Mises) and the stress-strain relations, yields a system of nonlinear ordinary differential equations for the coefficients. These equations are exactly solvable for Mode-III, and they have been solved in an approximate manner for Mode-I plane stress. The crack-line fields have been matched to appropriate elastic fields at the elastic-plastic boundary. For both Mode-III and Mode-I plane stress, the plastic strains, which depend on the elastodynamic stress intensity factor and the crack-tip speed, have been used in conjunction with the crack growth criterion of critical plastic strain, to determine the relation between the far-field stress level and the crack-tip speed.

1. Introduction

At high crack-tip speeds the mass density of a material affects the fields of stress and deformation in the vicinity of a propagating crack tip. For essentially brittle fracture, near-tip dynamic effects have been investigated extensively on the basis of linear elastic fracture mechanics. By now, several papers have reviewed the computation of elastodynamic stress intensity factors, and they have discussed dynamic effects on the fracture criterion of the balance of rates of energies, see Achenbach (1), Freund (2) and Kanninen (3). The combined effect of plastic deformation and mass density on near-tip fields has not yet received that much attention. This is not surprising, considering the difficulties that are encountered in the quasi-static analysis of fields near a growing crack in an elastic-plastic material.

For quasi-statically growing cracks the asymptotic structure of near-tip fields in elastic perfectly-plastic solids has been analyzed in considerable detail. A recent review by Rice (4) includes a general formulation, and it presents detailed results for isotropic materials of the Huber-Mises type. In general, the analytical near-tip results must be supplemented by numerical calculations to determine certain arbitrary functions that appear in the asymptotically valid near-tip results.

In recent papers, Achenbach and Dunayevsky (5) and Achenbach and Li (6) have constructed quasi-static solutions that are valid on the crack line, from the moving crack tip up to the moving elastic-plastic boundary.

These solutions were obtained for an elastic perfectly-plastic material of the Huber-Mises type by expanding all fields in powers of the distance, y, to the crack line. Substitution of the expansions in the equilibrium equations, the yield condition and the constitutive equations yields a system of simple ordinary differential equations for the coefficients of the expansions. As shown in (6), the resulting equations are exactly solvable for the Mode-III case, and they are solvable for the Mode-I plane-stress case if it is assumed that the cleavage stress is uniform on the crack line. By matching the relevant stress components and particle velocities to the dominant terms of appropriate elastic fields at the elastic-plastic boundary, the plastic strains on the crack line were computed in terms of the elastic stress intensity factor.

The literature on dynamic effects in the presence of elastic-plastic constitutive behavior is growing. Investigations of the asymptotic structure of the dynamic near-tip fields were presented by Slepyan (7) and Achenbach and Dunayevsky (8). Dynamic near-tip effects for a strain-hardening material were investigated by Achenbach and Kanninen (9) and Achenbach, Kanninen and Popelar (10) on the basis of J_2-flow theory and a bilinear effective stress-strain relation. For Mode-III crack propagation in an elastic perfectly-plastic material, exact crack-line solutions were obtained by Achenbach and Dunayevsky (11) and Freund and Douglas (12).

In the present paper the expansion technique of Achenbach and Li (6) is extended to the dynamic formulation, for rapid crack growth in Mode-III and in Mode-I plane stress. Systems of nonlinear ordinary differential

equations have been established which are valid for the transient case. Solutions have, however, been obtained only for the steady-state dynamic crack line fields. The equations for the Mode-III case can be solved rigorously in implicit form. An approximate approach which gives excellent results for the Mode-III case has, however, also been developed. The equations for Mode-I plane stress cannot be solved rigorously, but the approximate approach can be used to yield the steady-state dynamic cleavage strain on the crack line. The plastic strains on the crack line have been used in conjunction with the crack growth criterion of critical plastic strain to determine a relation between the far-field stress level and the crack-tip speed.

The geometry is shown in Fig. 1. The x_3-axis of a stationary coordinate system is parallel to the crack front, and x_1 points in the direction of crack growth. The position of the crack tip is defined by $x_1 = a(t)$. A moving coordinate system (x,y,z) is centered at the crack tip, with its axes parallel to the x_1,x_2 and x_3 axes.

2. Mode III Crack Propagation

In this Section an exact steady-state dynamic solution is derived which is valid on the crack line in the plastic loading zone ahead of the propagating crack tip.

In the moving coordinate system the equation of motion is

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = \rho \ddot{w} \tag{2.1}$$

where $w(x,y,t)$ is the anti-plane displacement, and the material time derivative is defined as

$$(\dot{\ }) = (\partial/\partial t) - \dot{a}\,(\partial/\partial x) \tag{2.2}$$

Here $\dot{a} = da/dt$ is the speed of the crack tip. The Huber-Mises yield condition requires

$$\tau_{xz}^2 + \tau_{yz}^2 = k^2 \,, \tag{2.3}$$

where k is the yield stress in pure shear. The strain rates are

$$\dot{\varepsilon}_{xz} = \frac{1}{2}\frac{\partial \dot{w}}{\partial x} \,, \qquad \dot{\varepsilon}_{yz} = \frac{1}{2}\frac{\partial \dot{w}}{\partial y} \tag{2.4a,b}$$

The strain rates are related to the stresses and the stress rates by

$$\dot{\varepsilon}_{xz} = \frac{\dot{\tau}_{xz}}{2\mu} + \dot{\Lambda}\tau_{xz} \,, \qquad \dot{\varepsilon}_{yz} = \frac{\dot{\tau}_{yz}}{2\mu} + \dot{\Lambda}\tau_{yz} \tag{2.5a,b}$$

In (2.5a,b) μ is the shear modulus and $\dot{\Lambda}$ is a positive function of time and the spatial coordinates.

Solution along the crack line.

In this paper we are interested in solutions along the crack line $y = 0$, $0 < x \leq x_p$, where $x = x_p$ defines the elastic-plastic boundary. Such solutions can be obtained by considering expansions with respect to y in the region $y/x \ll 1$:

$$\tau_{yz} = s_o(x,t) + s_2(x,t)y^2 + O(y^4) \tag{2.6}$$

$$\tau_{xz} = \tau_1(x,t)y + O(y^3) \tag{2.7}$$

$$\dot{w} = \dot{w}_1(x,t)y + O(y^3) \tag{2.8}$$

$$\dot{\Lambda} = \dot{\Lambda}_o(x,t) + \dot{\Lambda}_2(x,t)y^2 + O(y^4) \tag{2.9}$$

In (2.6)-(2.9) we have taken into account that τ_{yz} and $\dot{\Lambda}$ are symmetric with respect to y = 0, while τ_{xz} and $\dot{w}$ are antisymmetric. Substitution of (2.6)-(2.9) into (2.1), (2.3) and (2.5a,b), and collecting terms of the lowest orders in y yields

$$\frac{\partial \tau_1}{\partial x} + 2s_2 = \rho \ddot{w}_1 \tag{2.10}$$

$$s_o^2 = k^2 \ , \quad 2s_o s_2 + \tau_1^2 = 0 \tag{2.11a,b}$$

$$\frac{1}{2}\frac{\partial \dot{w}_1}{\partial x} = \frac{\dot{\tau}_1}{2\mu} + \dot{\Lambda}_o \tau_1, \quad \frac{1}{2}\dot{w}_1 = \dot{\Lambda}_o s_o \tag{2.12a,b}$$

It follows from (2.11a) that $s_o = k$. Elimination of s_2 from (2.10) and (2.11b) gives

$$\frac{\partial \tau_1}{\partial x} - \frac{\tau_1^2}{k} - \rho \ddot{w}_1 = 0 \tag{2.13}$$

Similarly, $\dot{\Lambda}_o$ can be eliminated from (2.12a) and (2.12b) to yield

$$\frac{\partial \dot{w}_1}{\partial x} - \frac{\dot{\tau}_1}{\mu} - \frac{1}{k}\dot{w}_1\tau_1 = 0 \tag{2.14}$$

Equations (2.13) and (2.14) define two coupled nonlinear partial differential equations. Analytical solutions to these equations, which would give the <u>transient</u> fields on the crack line, have not yet been obtained.

Equations (2.13) and (2.14) must be supplemented by conditions at the elastic-plastic boundary Σ. These have been discussed in some detail in Appendix A, where it was shown that for conditions which may be assumed to hold ahead of a propagating crack tip, the stresses are continuous at Σ, see Eq.(A.24a,b). From the impulse momentum relation (A.3) it follows that the particle velocity is then also continuous. Near the crack line at $x = x_p$ we can then write:

$$[[s_o]] = 0 \ , \quad [[\tau_1]] = 0 \text{ and } \quad [[\dot{w}_1]] = 0 \ , \tag{2.15a,b,c}$$

where the notation for discontinuities is defined by Eq.(A.1).

The governing equations for the quasi-static case follow by setting $\rho \equiv 0$. The resulting system of coupled nonlinear ordinary differential equations can be solved. The quasi-static solution for $\dot{w}_1$ has been given in Ref.(5).

For the steady-state case the material time derivative (2.2) reduces to

$$(\dot{\ }) = - \dot{a}\ (d/dx) \tag{2.16}$$

where $\dot{a}$ is now a constant crack tip speed. We define

$$\gamma_1 = \frac{dw_1}{dx} \ , \text{ and hence } \dot{w}_1 = - \dot{a}\gamma_1 \ , \tag{2.17a,b}$$

and we note that (2.14) and (2.13) then may be written as

$$\frac{d\gamma_1}{dx} - \frac{1}{\mu}\frac{d\tau_1}{dx} - \frac{1}{k}\gamma_1\tau_1 = 0 \tag{2.18}$$

$$\frac{d\tau_1}{dx} - \frac{\tau_1^2}{k} - \mu M^2 \frac{d\gamma_1}{dx} = 0 \ , \tag{2.19}$$

where the Mach number M is defined as

$$M = \mathring{a}/(\mu/\rho)^{\frac{1}{2}}. \tag{2.20}$$

For small values of y (i.e., $y/x \ll 1$) the plastic fields in the loading zone will be matched to the dominant terms of the elastic fields. In polar coordinates R,ψ centered at point E, and for small values of the angle ψ, the dominant terms of the solution on the elastic side of the elastic-plastic boundary are taken as

$$w \cong \left(\frac{R}{2\pi}\right)^{\frac{1}{2}} \frac{2}{\mu} K_{III} \frac{1}{2}\psi \ , \ \mu = \text{shear modulus} \tag{2.21}$$

$$\tau_{Rz} \cong \left(\frac{1}{2\pi R}\right)^{\frac{1}{2}} K_{III} \frac{1}{2}\psi \ , \quad \tau_{\psi z} \cong \left(\frac{1}{2\pi R}\right)^{\frac{1}{2}} K_{III} \tag{2.22a,b}$$

Here the elastic stress-intensity factor K_{III} depends on M. The angular dependence on M enters in higher order terms of ψ. It should be noted that the center of the elastic field is not taken to coincide with the crack tip. The center is located at a moving point E. The geometry is shown in Fig. 1.

Since τ_{yz} is continuous at $y = 0$, $x = x_p$, see Eq.(2.15a), we find

$$\left(\frac{1}{2\pi R_p}\right)^{\frac{1}{2}} K_{III} = k \quad , \quad \text{or} \quad R_p = (K_{III})^2/2\pi k^2 \tag{2.23a,b}$$

where $R = R_p$ defines the radius of curvature of the elastic-plastic boundary, at least for small values of ψ. Another condition is that τ_{Rz} (i.e., the shear stress in the R,ψ system) should be continuous at the elastic-plastic boundary. We find by using (2.6) and (2.7)

$$\tau_{RZ} = \tau_{xz} \cos\psi + \tau_{yz} \sin\psi \cong \tau_1 y + k\psi \tag{2.24}$$

Thus, by the use of (2.22a) and (2.23a)

$$\tau_1 y + k\psi = k\frac{1}{2}\psi \tag{2.25}$$

Since $\psi \cong y/R_p$, we obtain at $x = x_p$

$$\tau_1 = - k/2R_p \tag{2.26}$$

Continuity of γ_1 at $x = x_p$, which follows from (2.15c), yields

$$\gamma_1 y = \frac{\partial w}{\partial R} - \frac{\psi}{R}\frac{\partial w}{\partial \psi} = - \frac{1}{2}\psi k \tag{2.27}$$

or

$$\gamma_1 = - k/2\mu R_p \tag{2.28}$$

where (2.23a) has been used. For completeness we list the condition on the strain $\partial w/\partial y$ at $x = x_p$

$$\gamma_y = \frac{\partial w}{\partial y} = w_1 = \frac{k}{\mu} \tag{2.29}$$

Equations (2.18) and (2.19) can be solved rigorously, as shown in Appendix B. It is, however, of interest to note that an asymptotic solution for small values of x can be obtained with minimal effort. Let us consider solutions of the general form

$$\tau_1 \cong - \bar{\tau}_1/x \ , \quad \gamma_1 \cong \bar{\gamma}_1/x \tag{2.30a,b}$$

Substitution in (2.18) and (2.19) yields

$$\bar{\tau}_1 = k(1 \pm M) \ , \quad \bar{\gamma}_1 = \pm (1 \pm M)k/\mu M \tag{2.31a,b}$$

Since we must have $\gamma_1 < 0$, we discard the solution containing the plus signs. Hence

$$\tau_1 \cong -\frac{k(1-M)}{x}, \qquad \gamma_1 \cong -\frac{k}{\mu}\frac{1-M}{M}\frac{1}{x} \tag{2.32a,b}$$

Since $\gamma_1 = \partial\gamma_y/\partial x$, we also have

$$\gamma_y \cong -\frac{k}{\mu}\frac{1-M}{M}\,\ell n(x/x_p) \tag{2.33}$$

This solution is the same as the one derived earlier by Slepyan (7), see also Achenbach and Dunayevsky (8). Note that τ_1 reduces to the quasi-static solution as $M \to 0$. The strain γ_y has, however, not only the wrong behavior in x, but actually becomes singular in M.

As shown in Appendix B, the solution to Eqs.(2.18) and (2.19) which satisfies the boundary conditions (2.26) and (2.28) at $x = x_p$ is defined by the following equations:

$$\frac{\tau_1}{k/2R_p} = -\left(\frac{1-M}{1+M}\right)^{1/2M}\frac{M}{(1-M^2)^{\frac{1}{2}}}\,\frac{\left(-2F-\frac{1}{1+M}\right)^{(1-M)/2M}}{\left(2F+\frac{1}{1-M}\right)^{(1+M)/2M}} \tag{2.34}$$

where

$$F(\tau_1) = -\frac{\frac{1}{2}k}{(\tau_1)^2}\frac{d\tau_1}{dx} \tag{2.35}$$

$$\gamma_1 = -\frac{\tau_1}{\mu M^2}\left[-k(1-M^2)\,\frac{1}{\tau_1^2}\frac{d\tau_1}{dx} + 1\right] \tag{2.36}$$

Equation (2.34) gives F as a function of τ_1. Integration of (2.35) then yields

$$x = -\frac{1}{2}k\int_{-k/2R_p}^{\tau_1}\frac{d\xi}{\xi^2 F(\xi)} + x_p \tag{2.37}$$

Equation (2.37) yields τ_1 as a function of x and x_p. Substitution of the result in Eq.(2.36) yields γ_1. By letting $\tau_1 \to \infty$ in (2.37) we obtain a relation between x_p and R_p. From $\gamma_1 = \partial\gamma_y/\partial x$, we finally find

$$\gamma_y = \frac{k}{\mu} + \int_{x_p}^{x} \gamma_1 \, dx = \frac{k}{\mu} + \gamma_y^p \tag{2.38}$$

The strain γ_y obtained from (2.38) is the exact solution on the crack line. This solution is equivalent to the one obtained earlier by Dunayevsky and Achenbach (8) , and Freund and Douglas (12). It can be shown that for small x, Eq.(2.38) reduces to (2.31b). In the limit $M \to 0$, (2.38) reduces to the quasi-static solution

$$\frac{\mu}{k}\gamma_y = 1 - \ln(x/x_p) + \frac{1}{2}[\ln(x/x_p)]^2 \tag{2.39}$$

An explicit analytical expression for γ_y, albeit an approximate one, would be very useful for applications in conjunction with the crack growth criterion of a critical plastic strain. Another reason for an approximate approach to the Mode-III case is that the results can be tested by comparison with exact results. The same approach can then be used for the Mode-I plane-stress case, which is not amenable to an exact solution.

An approximate approach is suggested by the structure of Eqs.(2.18) and (2.19). If an acceptable approximation to τ_1 would be available a-priori, then (2.18) would simply be a linear ordinary differential equation for γ_1. A first approximation to τ_1 is suggested by (2.32a),

namely, $\tau_1 = -k(1-M)/x$. This expression has the correct limits at $M = 0$ and $M = 1$. A better result is obtained by adding a constant term

$$\tau_1 = - k(1-M)\left[\frac{1}{x} + \frac{M}{2x_p}\right] \tag{2.40}$$

The second term is chosen so that (2.19) is satisfied up to order $O(M)$ near $x = x_p$. Figure 2 shows a comparison between (2.40) and the exact result. Equation (2.40) can now be substituted in (2.18), and the resulting equation can be solved rigorously for γ_1. The strain $\gamma_y = k/\mu + \gamma_y^p$ then follows from (2.38). In anticipation of difficulties with the Mode-I case, we elect, however, to solve γ_1 by using a perturbation solution which ignores terms of order $O(M^2)$. The corresponding expression for γ_y is obtained as

$$\frac{\mu}{k}\gamma_y^p\left(\frac{x}{x_p}\right) = \frac{(1-M)(2-3M-M^2)}{M(2-M+M^2)}\left[\frac{1}{M}\left(\frac{x}{x_p}\right)^M - \frac{1}{M} - \frac{M(1-M)}{2(1+M)}\left(\frac{x}{x_p}\right)^{1+M} + \frac{M(1-M)}{2(1+M)}\right]$$
$$- \frac{1-M}{M}\,\ell n\left(\frac{x}{x_p}\right) + \frac{1}{2}(1-M)\left(\frac{x}{x_p} - 1\right) \tag{2.41}$$

It is of interest that (2.41) yields (2.33) in the limit $x \to 0$, while it yields the quasi-static solution (2.39) as $M \to 0$, provided that

$$M\,\ell n\left(\frac{x}{x_p}\right) << 1 \tag{2.42}$$

A comparison of (2.41) with the exact result is shown in Fig. 3.

Application of the boundary condition (2.26) to the approximate expression for τ_1, yields a relation between x_p and R_p. Subsequent use of (2.23b) gives

$$x_p = (1-M)(1+2M) \frac{1}{2\pi} (K_{III}/k)^2 \tag{2.43}$$

In Ref.(12), Freund and Douglas use numerical results of a finite element analysis to derive

$$x_p = (0.295 - 0.5M^2)(K_{III}/k)^2 \tag{2.44}$$

For arbitrary M we cannot obtain an exact analytical expression for x_p as a function of K_{III}/k from (2.34)-(2.37). However, as $M \to 0$, we find

$$x_p = (2-6M^2) \frac{1}{2\pi}(K_{III}/k)^2 \tag{2.45}$$

Figure 4 shows a comparison between (2.43), (2.44) and the exact relation, which follows from (2.34)-(2.37).

Finally, following Freund and Douglas (12) we apply the crack growth criterion of critical plastic strain to determine the value of K_{III} that would be required for crack growth at a given value of M. The crack-growth criterion, originally proposed by McClintock and Irwin (13), states that the crack will grow with (normalized) plastic strain $(\mu/k)\gamma_y^p = \gamma_f$ at $x = x_f$ on $y = 0$. For plastic strain below γ_f at $x = x_f$ the crack cannot grow. As discussed by Rice (14) the characteristic length x_f is related to K_c, the value of the Mode III stress intensity factor which is required to satisfy the fracture criterion for a <u>stationary</u> crack, by the relation

$$\pi x_f(\gamma_f+1) = (K_c/k)^2 \quad , \qquad \text{or } x_f = (K_c/k)^2/\pi(\gamma_f+1) \tag{2.46a,b}$$

We can now compute γ_f from (2.41) by substituting x_f for x. Subsequent elimination of x_p by the use of (2.43) yields

$$\gamma_f = \frac{\mu}{k}\, \gamma_y^p(\xi) \tag{2.47}$$

where the functional form of γ_y^p is given by (2.41), and the argument ξ is

$$\xi = 2(K_c/K_{III})^2/[(\gamma_f+1)(1-M)(2+M)] \tag{2.48}$$

For three values of γ_f, the relation between K_{III}/K_c and M given by (2.47) has been plotted in Fig. 5, and compared with the exact relation.

The elastodynamic stress intensity factor K_{III} is the dynamic factor. It is related to the corresponding quasi-static factor, see Ref.[1, p.35] by the relation

$$K_{III} = (1-M\ )^{\frac{1}{2}}(K_{III})_{qs} \tag{2.49}$$

Equation (2.49) implies that remote load, to attain a high crack tip speed is actually even higher than would follow from (2.47), because the external load is contained in $(K_{III})_{qs}$.

3. Mode-I Crack Propagation in Plane Stress

We consider a state of generalized plane stress, hence σ_z, σ_{xz} and σ_{yz} vanish identically. Relative to the moving coordinate system the equations of motion are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \ddot{u} \ , \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho \ddot{v} \tag{3.1a,b}$$

The Huber-Mises yield condition becomes

$$\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 = 3k^2 , \tag{3.2}$$

where k is as in Eq.(2.3). The strain rates are

$$\dot{\varepsilon}_x = \frac{\partial \dot{u}}{\partial x} , \quad \dot{\varepsilon}_y = \frac{\partial \dot{v}}{\partial y} , \quad \dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right) \tag{3.3a,b,c}$$

The strain rates are related to the stresses and stress rates by

$$\frac{\partial \dot{u}}{\partial x} = \frac{1}{E}(\dot{\sigma}_x - \nu \dot{\sigma}_y) + \frac{1}{3} \dot{\Lambda}(2\sigma_x - \sigma_y) \tag{3.4}$$

$$\frac{\partial \dot{v}}{\partial y} = \frac{1}{E}(\dot{\sigma}_y - \nu \dot{\sigma}_x) + \frac{1}{3} \dot{\Lambda}(2\sigma_y - \sigma_x) \tag{3.5}$$

$$\frac{1}{2}\left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right) = \frac{1+\nu}{E} \ \dot{\tau}_{xy} + \dot{\Lambda}\tau_{xy} \ , \tag{3.6}$$

where E and ν are Young's modulus and Poisson's ratio, respectively, and $\dot{\Lambda}$ is a positive function of time and the spatial coordinates.

Solution along the crack line

Analogously to (2.6)-(2.9) we consider

$$\sigma_x = p_o(x,t) + p_2(x,t)y^2 + p_4(x,t)y^4 + O(y^6) \tag{3.7}$$

$$\sigma_y = q_o(x,t) + q_2(x,t)y^2 + q_4(x,t)y^4 + O(y^6) \tag{3.8}$$

$$\tau_{xy} = s_1(x,t)y + s_3(x,t)y^3 + O(y^5) \tag{3.9}$$

$$\dot{u} = \dot{u}_o(x,t) + \dot{u}_2(x,t)y^2 + O(y^4) \tag{3.10}$$

$$\dot{v} = \dot{v}_1(x,t)y + \dot{v}_3(x,t)y^3 + O(y^5) \tag{3.11}$$

$$\dot{\Lambda} = \dot{\Lambda}_o(x,t) + \dot{\Lambda}_2(x,t)y^2 + O(y^4) \tag{3.12}$$

Here we have taken into account that σ_x, σ_y, u and $\dot{\Lambda}$ are symmetric with respect to y = 0, while τ_{xy} and $\dot{v}$ are antisymmetric. Substitution of (3.7)-(3.9) into (3.1a,b) and collecting terms of the same order in y yields

$$\frac{\partial p_o}{\partial x} + s_1 = \rho\ddot{u}_o\ , \quad \frac{\partial p_2}{\partial x} + 3s_3 = \rho\ddot{u}_2 \tag{3.13a,b}$$

$$\frac{\partial s_1}{\partial x} + 2q_2 = \rho\ddot{v}_1\ , \quad \frac{\partial s_3}{\partial x} + 4q_4 = \rho\ddot{v}_3 \tag{3.14a,b}$$

Substitution of (3.7)-(3.9) into the yield condition (3.2) yields by the same procedure

$$p_o^2 + q_o^2 - p_o q_o = 3k^2 \tag{3.15}$$

$$(2p_o - q_o)p_2 + (2q_o - p_o)q_2 + 3s_1^2 = 0 \tag{3.16}$$

$$p_2^2 + (2p_o - q_o)p_4 + q_2^2 + (2q_o - p_o)q_4 - p_2 q_2 + 6s_1 s_3 = 0 \tag{3.17}$$

Another 5 equations are obtained by using (2.16) and (3.7)-(3.10) in (3.4)-(3.6). These equations have been listed as Eqs.(21)-(25) by

Achenbach and Li (6), and they are not reproduced here.

At this stage we have 14 unknowns and 12 equations. Clearly, the system cannot be solved without further simplifying assumptions. For the quasi-static problem (i.e., $\rho \equiv 0$), one assumption, namely that q_o = constant, suffices to produce a solvable system of equations, as shown by Achenbach and Li (6). For reference purposes we state the coefficients for the quasi-static stresses obtained in (6)

$$p_o = k \; , \; p_2 = -\frac{3}{2}\frac{k}{x^2} \tag{3.18a,b}$$

$$q_o = 2k \; , \; q_2 = 0 \tag{3.19a,b}$$

$$s_1 = 0 \; , \quad s_3 = -\frac{k}{x^3} \tag{3.20a,b}$$

An approximate solution can be obtained for the steady-state dynamic problem, when $(\dot{\ }) = -\dot{a}d/dx$. Equations (3.13)-(3.14) then become

$$\frac{dp_o}{dx} + s_1 = EM^2 \frac{d^2u_o}{dx^2} \; , \qquad \frac{dp_2}{dx} + 3s_3 = EM^2 \frac{d^2u_2}{dx^2} \tag{3.21a,b}$$

$$\frac{ds_1}{dx} + 2q_2 = EM^2 \frac{d^2v_1}{dx^2} \; , \qquad \frac{ds_3}{dx} + 4q_4 = EM^2 \frac{d^2v_3}{dx^2} \tag{3.22a,b}$$

where M is defined as

$$M = \dot{a}/(E/\rho)^{\frac{1}{2}} \tag{3.23}$$

The equations that can be obtained from (3.4)-(3.6) now become

$$\frac{d^2u_o}{dx^2} = \frac{1}{E}\left(\frac{dp_o}{dx} - \nu\frac{dq_o}{dx}\right) + \frac{1}{3}\frac{d\Lambda_o}{dx}(2p_o - q_o) \tag{3.24}$$

$$\frac{d^2u_2}{dx^2} = \frac{1}{E}\left(\frac{dp_2}{dx} - \nu\,\frac{dq_2}{dx}\right) + \frac{1}{3}\,\frac{d\Lambda_o}{dx}(2p_2 - q_2) + \frac{1}{3}\,\frac{d\Lambda_2}{dx}(2p_o - q_o) \tag{3.25}$$

$$\frac{dv_1}{dx} = \frac{1}{E}\left(\frac{dq_o}{dx} - \nu\,\frac{dp_o}{dx}\right) + \frac{1}{3}\,\frac{d\Lambda_o}{dx}(2q_o - p_o) \tag{3.26}$$

$$3\,\frac{dv_3}{dx} = \frac{1}{E}\left(\frac{dq_2}{dx} - \nu\,\frac{dp_2}{dx}\right) + \frac{1}{3}\,\frac{d\Lambda_o}{dx}(2q_2 - p_2) + \frac{1}{3}\,\frac{d\Lambda_2}{dx}(2q_o - p_o) \tag{3.27}$$

$$\frac{du_2}{dx} + \frac{1}{2}\,\frac{d^2v_1}{dx^2} = \frac{1+\nu}{E}\,\frac{ds_1}{dx} + \frac{d\Lambda_o}{dx}\,s_1 \tag{3.28}$$

To determine a solution to Eqs.(3.15)-(3.17), (3.21)-(3.22) and (3.24)-(3.28), we start by making the same assumption as for the quasi-static case, namely, that q_o = constant. A second assumption is that $2p_o - q_o = \varepsilon$, where $\varepsilon = \varepsilon(M)$, but $\varepsilon \ll k$. It then follows from (3.15) that

$$p_o = k + O(\varepsilon),\quad q_o = 2k + O(\varepsilon^2),\quad 2q_o - p_o = 3k + O(\varepsilon),\quad 2p_o - q_o = O(\varepsilon) \tag{3.29a,b,c,d}$$

Since both p_o and q_o are constant, Eq.(3.26) implies that $d\Lambda_o/dx = (1/k)dv_1/dx + O(\varepsilon)$, and it subsequently follows from (3.24) that $d^2u_o/dx^2 = O(\varepsilon)$. Substitution of these results in (3.21a) gives $s_1 = O(\varepsilon M^2)$, while (3.22a) gives $q_2 = O(M^2)$. Next, we conclude from (3.16) that $\varepsilon p_2 + 3kq_2 + O(\varepsilon^2 M^4) = 0$, which implies that $\varepsilon = O(M^2)$. Application of the preceding results to (3.17) gives $q_4 = -(1/3k)p_2^2 + O(M^2)$. Substitution of the latter result in (3.22b), and then in (3.21b) yields

$$\frac{d^2p_2}{dx^2} + \frac{4}{k}\,p_2^2 = EM^2\frac{d^3u_2}{dx^3} + O(M^2) \tag{3.30}$$

Note that the inertia term has not been neglected in this equation, since it provides a coupling with equations for u_2. Substitution of $s_1 = O(\epsilon M^2) = O(M^4)$ into (3.28) yields

$$\frac{du_2}{dx} + \frac{1}{2}\frac{d^2v_1}{dx^2} = O(M^4) \tag{3.31}$$

Finally, by using (3.31), as well as (3.29) and $d\Lambda_o/dx = (1/k)dv_1/dx$, Eq.(3.25) gives

$$\frac{1}{2}\frac{d^3v_1}{dx^3} + \frac{1}{E}\frac{dp_2}{dx} + \frac{2}{3k}\frac{dv_1}{dx}\,p_2 = O(M^2) \tag{3.32}$$

In a further reduction we ignore the terms of $O(M^2)$, and we eliminate u_2 by the use of (3.31), to obtain

$$\frac{d^2p_2}{dx^2} + \frac{4}{k}\,p_2^2 + \frac{1}{2}\,EM^2\,\frac{d^3v_{1x}}{dx^3} = 0 \tag{3.33}$$

$$\frac{1}{2}\frac{d^2v_{1x}}{dx^2} + \frac{2}{3k}\,p_2\,v_{1x} + \frac{1}{E}\frac{dp_2}{dx} = 0\ , \tag{3.34}$$

where

$$v_{1x} = dv_1/dx \tag{3.35}$$

Equations (3.33)-(3.34) will be used to analyze the Mode-I plane-stress fields.

The solutions for v_{1x} and p_2 must satisfy certain conditions at the elastic plastic boundary Σ. In Appendix A it was shown that for conditions which may be assumed to hold ahead of a propagating crack tip, the stresses are continuous at Σ , see Eq.(A.31). From the impulse momentum relation (A.3) it then follows that the particle velocity is also continuous at Σ.

For $y/x \ll 1$ the plastic fields in the loading zone will now be matched to the dominant terms of the elastic field. In polar coordinates R, ψ, centered at point E, the elastic field for small values of ψ is taken as

$$\sigma_x = \left(\frac{1}{2\pi R}\right)^{\frac{1}{2}} K_I\left\{\left(1 - \frac{\eta}{2R}\right) - \left(\frac{7}{8} - \frac{9\eta}{16R}\right)\psi^2\right\} \tag{3.36}$$

$$\sigma_y = \left(\frac{1}{2\pi R}\right)^{\frac{1}{2}} K_I\left\{\left(1 + \frac{\eta}{2R}\right) + \left(\frac{5}{8} - \frac{9\eta}{16R}\right)\psi^2\right\} \tag{3.37}$$

$$\tau_{xy} = \left(\frac{1}{2\pi R}\right)^{\frac{1}{2}} K_I\left(\frac{1}{2} - \frac{3\eta}{4R}\right)\psi \tag{3.38}$$

$$u = \left(\frac{R}{2\pi}\right)^{\frac{1}{2}} \frac{1}{2\mu} K_I\left\{\left(\kappa - 1 + \frac{\eta}{R}\right) + \frac{1}{8}\left(5 - \kappa - \frac{\eta}{R}\right)\psi^2\right\} \tag{3.39}$$

$$v = \left(\frac{R}{2\pi}\right)^{\frac{1}{2}} \frac{1}{2\mu} K_I\left(\kappa - 1 + \frac{\eta}{2R}\right)\psi , \tag{3.40}$$

where $\kappa = (3-\nu)/(1+\nu)$. This elastic field has one more parameter, namely η, than the usual elastic crack-tip field. Equations (3.36) - (3.40) actually correspond to the field for a notch with $\frac{1}{2}\eta$ as its tip-radius of curvature, see Creager and Paris (15). The elastic stress-intensity factor K_I depends on M, but the angular dependence on M enters in higher order terms of ψ. The center of the elastic field is located at the moving point E.

Since the stresses σ_x and σ_y are continuous at $\psi = 0$, we find from (3.29a,b) and (3.36)-(3.37)

$$\left(\frac{1}{2\pi R_p}\right)^{\frac{1}{2}} K_I\left(1 - \frac{\eta}{2R_p}\right) = k, \quad \left(\frac{1}{2\pi R_p}\right)^{\frac{1}{2}} K_I\left(1 + \frac{\eta}{2R_p}\right) = 2k \quad , \tag{3.41a,b}$$

where $R = R_p$ defines the radius of curvature of the elastic-plastic boundary, at least for small values of ψ. From (3.41a,b) we obtain $\eta/R_p = 2/3$. Substitution of this result in (3.41a) yields

$$\left(\frac{1}{2\pi R_p}\right)^{\frac{1}{2}} K_I = \frac{3}{2}k\ , \quad \text{or} \quad R_p = (4/9)(K_I/k)^2/2\pi \tag{3.42a,b}$$

For small values of y/x and ψ we next consider the continuity of $\tau_{R\psi}$ and σ_R. We use

$$\tau_{R\psi} = (\cos^2\psi - \sin^2\psi)\ \sigma_{xy} + (\sigma_y - \sigma_x)\sin\psi\cos\psi \tag{3.43}$$

$$\sigma_R = 2\sigma_{xy}\sin\psi\cos\psi + \sigma_x\cos^2\psi + \sigma_y\sin^2\psi\ , \tag{3.44}$$

in conjunction with (3.36)-(3.38) and the stresses in the plastic zone. It may be verified that $\tau_{R\psi}$ is continuous to first order in ψ . The stress σ_R is continuous to order unity by virtue of Eq.(3.42). Collecting terms to order y^2 and ψ^2 yields the relation

$$x = x_p: \quad p_2 = -\frac{3k}{4R_p^2} \tag{3.45}$$

Next, we consider the continuity of the particle velocity at small values of ψ. We use

$$\dot{u}_\psi = \dot{v}\cos\psi - \dot{u}\sin\psi \tag{3.46}$$

$$\dot{u}_R = \dot{v}\sin\psi + \dot{u}\cos\psi \tag{3.47}$$

Since $[\dot{u}_\psi] = 0$ and $[\dot{u}_R] = 0$, we find from (3.46)-(3.47)

$$[\dot{v}]\cos\psi - [\dot{u}]\sin\psi = 0 \tag{3.48}$$

$$[\dot{v}]\sin\psi + [\dot{u}]\cos\psi = 0 \tag{3.49}$$

Substitution of (3.11)-(3.12) into (3.48)-(3.49) and collecting terms of the same order in y yields

$$[\dot{u}_o] = 0 \ , \quad [\dot{v}_1] = 0 \quad [\dot{u}_2] = 0 \qquad \text{(3.50a,b,c)}$$

By the use of (3.39)-(3.40), we then obtain

$$x=x_p: \qquad \dot{v}_1 = \frac{3k}{2ER_p}\dot{a} \ , \quad \dot{u}_2 = \frac{3(2+\nu)k}{8ER_p^2}\dot{a} \ , \qquad \text{(3.51a,b)}$$

where (2.16) has been used. For the steady-state problem (3.51a,b) imply:

$$x=x_p: \qquad v_{1x} = -3k/2ER_p, \quad \frac{dv_{1x}}{dx} = 3(2+\nu)k/4ER_p^2 \qquad \text{(3.52a,b)}$$

It appears to be difficult to solve (3.33) and (3.34) rigorously. Just as for the Mode-III case, an asymptotic solution for small values of x can, however, easily be obtained by considering solutions of the form

$$p_2 \cong P_2/x^2, \quad v_{1x} \cong V_{1x}/x \qquad \text{(3.53a,b)}$$

The appropriate constants follow from (3.33)-(3.34) as

$$P_2 = -(3k/2)(1-M), \qquad V_{1x} = (-3k/E)(1-M)/M \qquad \text{(3.54a,b)}$$

The corresponding strain ε_y is

$$\varepsilon_y \cong -3\frac{k}{E}\,\frac{1-M}{M}\,\ln\left(\frac{x}{x_p}\right) \tag{3.55}$$

In the limit $M \to 0$, p_2 reduces to the quasi-static solution given by Eq.(3.18b), but ε_y becomes singular.

The similarities in the structure of the equations for the Mode-III and Mode-I plane stress cases suggests an approximate approach to (3.33)-(3.34) similar to the one used for solving (2.18) and (2.19). Thus, if an acceptable approximation to p_2 would be available a-priori, then (3.34) would be a linear ordinary differential equation for v_{1x}. A first approximation to p_2 is provided by the asymptotic expression (3.52a). This expression has the correct limits at $M = 0$ (quasi-static case) and $M = 1$. It may, however, be expected that a better approximation will be obtained by adding a constant term, and use

$$p_2 = -\frac{3}{2}k(1-M)\left[\frac{1}{x^2} + \frac{M}{2x_p^2}\right] \tag{3.56}$$

The second term is chosen so that (3.33) is satisfied up to order O(M) near $x = x_p$. It is noted that (3.56) is completely analogous to (2.40). Substitution of (3.56) into (3.34) yields

$$\frac{1}{2}\frac{d^2v_{1x}}{dx^2} - (1-M)\left[\frac{1}{x^2} + \frac{M}{2x_p^2}\right]v_{1x} = -\frac{3k(1-M)}{Ex^3} \tag{3.57}$$

An expression for x_p is obtained by enforcing the condition (3.45) on p_2:

$$x_p = [(1-M)(2+M)]^{\frac{1}{2}}R_p \tag{3.58}$$

A solution to (3.57) is obtained by using a perturbation solution which neglects terms of order $O(M^2)$. By integrating the result, the strain $\varepsilon_y = v_1$ is obtained as

$$\varepsilon_y = (\varepsilon_y)_{PB} + \varepsilon_y^p(x/x_p,M), \quad \text{where} \quad (\varepsilon_y)_{PB} = (k/E)(2-\nu) \tag{3.59a,b}$$

$$\varepsilon_y^p(x/x_p,M) = \frac{k}{E}\left\{\frac{\Delta_1}{\Delta}\left[\frac{1}{1+\alpha_1}\left(\left(\frac{x}{x_p}\right)^{1+\alpha_1}-1\right)+\frac{\beta_1}{3+\alpha_1}\left(\left(\frac{x}{x_p}\right)^{3+\alpha_1}-1\right)\right]\right.$$

$$\left.+\frac{\Delta_2}{\Delta}\left[\frac{1}{1+\alpha_2}\left(\left(\frac{x}{x_p}\right)^{1+\alpha_2}-1\right)+\frac{\beta_2}{3+\alpha_2}\left(\left(\frac{x}{x_p}\right)^{3+\alpha_2}-1\right)\right]-\frac{3(1-M)}{M}\left[\ln\left(\frac{x}{x_p}\right)-\frac{M}{4}\left(\left(\frac{x}{x_p}\right)^2-1\right)\right]\right\} \tag{3.60}$$

Also

$$\alpha_1 = \frac{1}{2}[1 + (9-8M)^{1/2}], \quad \alpha_2 = \frac{1}{2}[1 - (9-8M)^{1/2}] \tag{3.61}$$

$$\beta_1 = \frac{M(1-M)}{2[2+(9-8M)^{1/2}]}, \quad \beta_2 = \frac{M(1-M)}{2[2-(9-8M)^{1/2}]} \tag{3.62a,b}$$

$$\Delta_1 = \frac{3}{2}\left[\frac{(1-M)(2-M)}{M} - \frac{x_p}{R_p}\right](\alpha_2+\alpha_2\beta_2+2\beta_2) + \frac{3}{2}\left[\frac{(1-M)(2+M)}{M} - \frac{(2+\nu)}{2}\left(\frac{x_p}{R_p}\right)^2\right](1+\beta_2) \tag{3.63}$$

$$\Delta_2 = -\frac{3}{2}\left[\frac{(1-M)(2+M)}{M} - \frac{2+\nu}{2}\left(\frac{x_p}{R_p}\right)^2\right](1+\beta_1) - \frac{3}{2}\left[\frac{(1-M)(2-M)}{M} - \frac{x_p}{R_p}\right](\alpha_1+\alpha_1\beta_1+2\beta_1) \tag{3.64}$$

$$\Delta = (\alpha_2-\alpha_1)(1+\beta_1+\beta_2+\beta_1\beta_2) + 2(\beta_2-\beta_1) \tag{3.65}$$

Here x_p/R_p is given by (3.58)

Equation (3.60) reduces to the quasi-static solution, which has been given by Achenbach and Li (6,Eq.(64), provided that $M\ln(x/x_p) \ll 1$. In the limit $x \to 0$, (3.59) reduces to (3.55).

Numerical results for ε_y as given by (3.59) are shown in Fig. 6, for $\nu = 0.3$ and $M = 0.1, 0.3, 0.5$. The quasi-static solution has also been shown in Fig. 6.

Finally, just as for the Mode-III case, we apply the crack growth criterion of critical plastic strain to determine the value of K_I that would be required for crack growth at a given value of M. For a stationary crack the quasi-static plastic strain follows from the results of Refs.(5) and (6) as

$$\epsilon_y^p(x/x_p) = \frac{k}{E}\left\{B_2\left(\frac{x_p}{x}\right) - \frac{1}{2}\, C_2\left(\frac{x}{x_p}\right)^2 - \frac{3}{4}(2+\nu)\right\} , \tag{3.66}$$

where

$$B_2 = \frac{1}{16}\,[(\kappa+5) + (\kappa+1)\sqrt{2}](E/\mu) \tag{3.67}$$

$$C_2 = \frac{1}{16}\,[-(\kappa+5) + (\kappa+1)2\sqrt{2}](E/\mu) \tag{3.68}$$

The constant κ is defined as $\kappa = (3-\nu)/(1+\nu)$. Note that ϵ_y^p properly vanishes at the elastic-plastic boundary.

Now suppose that the normalized critical strain

$$\epsilon_f = (\epsilon_y^p)_{cr}/(\epsilon_y)_{PB} \tag{3.69}$$

is reached at $x = x_f$, for a value of $K_I = K_{Ic}$. The corresponding value of x_p is given by Ref.(6,Eq.(57)) as

$$x_{pc} = \frac{2\sqrt{2}}{9}\,\frac{1}{\pi}\,(K_{Ic}/k)^2 \tag{3.70}$$

A cubic equation for x_f/x_{pc} follows from (3.66). The relevant real-valued root is

$$x_f/x_{pc} = S \tag{3.71}$$

where

$$S = (B_2/C_2+A)^{1/3} - (A-B_2/C_2)^{\frac{1}{2}} \tag{3.72}$$

$$A = \left\{ (B_2/C_2)^2 + \left[\frac{1-\frac{1}{2}\nu + (E/k)\varepsilon_f/3}{C_2}\right]^3 \right\}^{\frac{1}{2}} \tag{3.73}$$

Next we turn to (3.60), and we compute $\varepsilon_y^p(x/x_p,M)$ at $\xi = x_f/x_p$. Since x_p is given by (3.58) we have

$$\xi = \frac{S\sqrt{2}}{(1-M)^{\frac{1}{2}}(2+M)^{\frac{1}{2}}} (K_{Ic}/K_I) \quad , \tag{3.74}$$

where (3.42b) has also been used. The crack growth criterion now yields

$$\varepsilon_f = \varepsilon_y^P(\xi,M)/(\varepsilon_y)_{PB} \; , \tag{3.75}$$

where the function form of ε_y^p is given by (3.60). Equation (3.75) has been used to plot K_I/K_{Ic} versus M for three values of ε_f in Fig. 7.

Appendix A

Conditions at a Fast-Moving Surface of Strong Discontinuity

In a recent paper Drugan and Rice (16) have shown that all stress components are continuous across a quasi-statically moving surface of strong discontinuity in an elastic-plastic solid. They also showed that the only components of strain which may suffer discontinuities across such a surface are the plastic components which do not deform elements in the plane of the surface, and these strains may be discontinuous only if the stress state at the moving surface meets specific conditions.

In this Appendix we attempt to extend some of the results of (16), to the case that the surface moves so fast that dynamic effects must be taken into account. The results will serve to establish the conditions at the leading edge of the elastic-plastic boundary of the plastic loading zone ahead of a rapidly propagating crack tip, particularly in the immediate vicinity of the crack line.

The propagating elastic-plastic boundary is denoted by Σ, see Fig. 1. The surface propagates with velocity V in the direction of the normal ξ_1. The coordinate system ξ_1, ξ_2, ξ_3 moves with the surface Σ. A discontinuity of a field quantity, say $g(\xi_1,\xi_2, \xi_3,t)$ is denoted in the usual manner by

$$[[g]] \equiv g^+ - g^- , \qquad \text{(A.1)}$$

where

$$g^{\pm} \equiv \lim_{\Delta\to 0} g(\xi_1,\xi_2,\xi_3,t_a \mp \Delta) \qquad \text{(A.2)}$$

where $\Delta \geq 0$, and t_a is the time at which Σ arrives at a particular material point. In the sequel Latin indices i,j,k have range 1,2,3. Greek indices α,β have range 2,3 only and thus refer to tensor components in planes parallel to planes that are tangential to Σ.

The impulse-momentum relation yields across Σ:

$$[[\sigma_{1j}]] = -\rho V\ [[\dot{u}_j]] \tag{A.3}$$

where ρ is the mass density. By virtue of displacement continuity we have

$$[[u_i]] = 0 \tag{A.4}$$

It follows from (A.4) that, see e.g. Hill (17),

$$[[\partial u_i/\partial \xi_\alpha]] = 0 \tag{A.5}$$

$$[[\dot{u}_j]] = -V[[\partial u_j/\partial \xi_1]] \tag{A.6}$$

By combining (A.3) and (A.6) there results

$$[[\sigma_{1j}]] = \rho V^2[[\partial u_j/\partial \xi_1]] \tag{A.7}$$

Equations (A.5) - (A.7) have some implications for discontinuities in the components of the small-strain tensor, $\varepsilon_{ij} = \frac{1}{2}(\partial u_i/\partial \xi_j + \partial u_j/\partial \xi_i)$. We find

$$[[\varepsilon_{11}]] = [[\partial u_1/\partial \xi_1]] = [[\sigma_{11}]]/\rho V^2 \tag{A.8}$$

$$[[\varepsilon_{1\alpha}]] = \frac{1}{2}\ [[\partial u_\alpha/\partial \xi_1]] = \frac{1}{2}[[\sigma_{1\alpha}]]/\rho V^2 \tag{A.9}$$

$$[[\varepsilon_{\alpha\beta}]] = 0 \tag{A.10}$$

The total strain is taken to be the sum of elastic and plastic parts

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij} \ , \tag{A.11}$$

where

$$\varepsilon^e_{ij} = \frac{1}{2\mu}\,\sigma_{ij} - \frac{\nu}{E}(\sigma_{kk})\delta_{ij} \tag{A.12}$$

By combining (A.8)-(A.10) with (A.11)-(A.12) we find

$$[[\varepsilon^p_{11}]] = \left(\frac{1}{\rho V^2} - \frac{1}{E}\right)[[\sigma_{11}]] + \frac{\nu}{E}([[\sigma_{22}]] + [[\sigma_{33}]]) \tag{A.13}$$

$$[[\varepsilon^p_{1\alpha}]] = \frac{1}{2}\left(\frac{1}{\rho V^2} - \frac{1}{\mu}\right)[[\sigma_{1\alpha}]] \tag{A.14}$$

$$[[\varepsilon^p_{\alpha\beta}]] = -\frac{1}{2\mu}\,[[\sigma_{\alpha\beta}]] + \frac{\nu}{E}\,[[\sigma_{kk}]]\delta_{\alpha\beta} \tag{A.15}$$

Plastic deformation is assumed to obey the maximum plastic work inequality

$$(\sigma_{ij} - \sigma^o_{ij})d\varepsilon^p_{ij} \geq 0 \tag{A.16}$$

where σ_{ij} is the stress state (at yield) corresponding to the plastic strain increment $d\varepsilon^p_{ij}$, and σ^o_{ij} is any other stress state which is at or below yield. Following (16) we integrate (A.16) for a material point during passage of the discontinuity surface Σ to obtain

$$W^p - \int_{\varepsilon^{p+}_{ij}}^{\varepsilon^{p-}_{ij}} \sigma^o_{ij}d\varepsilon^p_{ij} \geq 0 \tag{A.17}$$

where σ^{o}_{ij} is understood to be a stress state at or below yield for all states along the strain path from $\underset{\sim}{\varepsilon}^{+}$ to $\underset{\sim}{\varepsilon}^{-}$. In the inequality (A.17), W^{p} is the plastic work accumulated discontinuously at a material point due to the passage of Σ:

$$W^{p} \equiv \int_{\varepsilon^{p+}_{ij}}^{\varepsilon^{p-}_{ij}} \sigma_{ij} d\varepsilon^{p}_{ij} \tag{A.18}$$

Subsequent considerations are for the special cases of anti-plane shear and generalized plane stress.

Anti-plane shear. This case is defined by $u_3 = u_3(\xi_1, \xi_2, t) \neq 0$, $u_1 = u_2 \equiv 0$. The relevant relations between the strain and stress increments across Σ follow from (A.14) and (A.15) as

$$d\varepsilon^{p}_{13} = \frac{1}{2}\left(\frac{1}{\rho V^2} - \frac{1}{\mu}\right) d\sigma_{13} \tag{A.19}$$

$$d\varepsilon^{p}_{23} = -\frac{1}{2\mu} d\sigma_{23} \tag{A.20}$$

Substitution of (A.19) and (A.20) into (A.18) yields

$$W^{p} = - \quad (\sigma^{+}_{13} + \sigma^{-}_{13})[[\varepsilon^{p}_{13}]] - \quad (\sigma^{+}_{23} + \sigma^{-}_{23})[[\varepsilon^{p}_{23}]] \tag{A.21}$$

For σ^{o}_{ij} we now choose

$$\sigma^{o}_{13} = \sigma^{-}_{13}, \quad \sigma^{o}_{23} = \sigma^{+}_{23}. \tag{A.22}$$

The inequality (A.17) then yields

$$-\frac{1}{2}\left(\frac{1}{\rho V^2} - \frac{1}{\mu}\right)[[\sigma_{13}]]^2 - \frac{1}{2\mu}[[\sigma_{23}]]^2 \geq 0 \tag{A.23}$$

If we restrict our attention to the sub-sonic case, for which $\rho V^2/\mu < 1$, it is evident that (A.23) can be satisfied only by

$$[[\sigma_{13}]] = 0 \text{ and } [[\sigma_{23}]] = 0 \tag{A.24a,b}$$

It remains to verify that the stress state (A.22a,b) is sub-yield. We should have $(\sigma_{13}^-)^2 + (\sigma_{23}^+)^2 \leq k^2$. Since the yield condition is satisfied at the - side of Σ, we do have $(\sigma_{13}^-)^2 + (\sigma_{23}^-)^2 = k^2$. By eliminating $(\sigma_{13}^-)^2$, the requirement that the stress state (A.22a,b) is sub-yield may then be written as

$$(\sigma_{23}^+)^2 - (\sigma_{23}^-)^2 \leq 0 \tag{A.25}$$

For example, (A.25) is satisfied if

$$\sigma_{23}^+ > 0 \ , \ \sigma_{23}^- > 0 \text{ and } \sigma_{23}^- \geq \sigma_{23}^+ \tag{A.26}$$

This is the case that generally applies at an elastic-plastic boundary ahead of a crack tip propagating in Mode-III.

On the crack line $\sigma_{13}^+ = \sigma_{13}^- \equiv 0$, and, the expression for W^p given by (A.21) simplifies to $W^p = - \quad (\sigma_{23}^+ + \sigma_{23}^-)[[\epsilon_{23}^p]]$. By taking $\sigma_{23}^o = \sigma_{23}^+$ (which is by definition sub-yield), we then obtain that $-(1/2\mu)[[\sigma_{23}]]^2 \geq 0$, which implies that on the crack line

$$[[\sigma_{23}]] = 0 \ , \tag{A.27}$$

without additional conditions.

Generalized plane stress. This case is defined by

$\sigma_{33} \equiv 0$, $\sigma_{22}(\xi_1,\xi_2,t) \neq 0$, $\sigma_{11}(\xi_1,\xi_2,t) \neq 0$, $\sigma_{12} = \sigma_{21}(\xi_1,\xi_2,t) \neq 0$.

The relevant relations between the strain and stress increments follow from (A.13) and (A.14) as

$$d\varepsilon^p_{11} = (\frac{1}{\rho V^2} - \frac{1}{E})\, d\sigma_{11} + \frac{\nu}{E}\, d\sigma_{22} \tag{A.28}$$

$$d\varepsilon^p_{22} = -\frac{1}{E}\, d\sigma_{22} + \frac{\nu}{E}\, d\sigma_{11} \tag{A.29}$$

$$d\varepsilon^p_{12} = \frac{1}{2}\,(\frac{1}{\rho V^2} - \frac{1}{\mu})\, d\sigma_{12} \tag{A.30}$$

Substitution of (A.28) - (A.30) into (A.18) yields

$$W^p = -\frac{1}{2}(\sigma^+_{11}+\sigma^-_{11})\;[[\varepsilon^p_{11}]] - \frac{1}{2}(\sigma^+_{22}+\sigma^-_{22})[[\varepsilon^p_{22}]] - (\sigma^+_{12}+\sigma^-_{12})[[\varepsilon^p_{12}]] \tag{A.31}$$

For σ^o_{ij} we choose

$$\sigma^o_{11} = \sigma^-_{11},\ \sigma^o_{22} = \sigma^+_{22}\ ,\ \sigma^o_{12} = \sigma^-_{12} \tag{A.32}$$

The inequality (A.17) then yields

$$-\frac{1}{2}(\frac{1}{\rho V^2} - \frac{1}{E})[[\sigma_{11}]]^2 - \frac{1}{2}\,\frac{1}{E}[[\sigma_{22}]]^2 - \frac{1}{2}\,(\frac{1}{\rho V^2} - \frac{1}{\mu})[[\sigma_{12}]]^2 \geq 0 \tag{A.33}$$

If we restrict our attention to $\rho V^2/\mu < 1$, then (A.33) can be satisfied only if

$$[[\sigma_{11}]] = 0\ ,\ [[\sigma_{22}]] = 0 \quad \text{and} \quad [[\sigma_{12}]] = 0 \tag{A.34a,b,c}$$

The results (A.34a,b,c) hold if the stress state (A.32) is indeed sub-yield, i.e., if

$$(\sigma^-_{11})^2 + (\sigma^+_{22})^2 - \sigma^-_{11}\sigma^+_{22} + 3(\sigma^-_{12})^2 \leq 3k^2 \tag{A.35}$$

Since the stress state σ^-_{11}, σ^-_{22} and σ^-_{12} satisfies the yield condition, $(\sigma^-_{11})^2 + 3(\sigma^-_{12})^2$ can be expressed in terms of $(\sigma^-_{22})^2$ and $\sigma^-_{11}\sigma^-_{22}$. Substitution of that result into (A.35), reduces that condition to

$$(\sigma_{22}^{+} + \sigma_{22}^{-} - \sigma_{11}^{-})[[\sigma_{22}]] \leq 0 \qquad \text{(A.36)}$$

This equation is satisfied if either

$$[[\sigma_{22}]] \leq 0 \text{ and } \sigma_{11}^{-} \leq \sigma_{22}^{+} + \sigma_{22}^{-} \qquad \text{(A.37a,b)}$$

or

$$[[\sigma_{22}]] \geq 0 \quad \text{and} \quad \sigma_{11}^{-} \geq \sigma_{22}^{+} + \sigma_{22}^{-} \qquad \text{(A.38a,b)}$$

Of interest in the present paper are discontinuities across the elastic-plastic boundary near the crack line, for Mode-I crack propagation in generalized plane stress. For that case we have near the crack line $\sigma_{22}^{+} > 0$ and $\sigma_{22}^{-} > 0$. We also have $\sigma_{11}^{-} \sim k$ and $\sigma_{22}^{-} \sim 2k$. Hence (A.37b) is satisfied. We will generally also have that $\sigma_{22}^{+} \leq \sigma_{22}^{-}$, hence $[[\sigma_{22}]] \leq 0$, and (A.37a) is satisfied. Thus, (A.32) is an acceptable sub-yield stress state, and the results (A.34a,b,c) are valid. Note that on the crack line $\sigma_{12} \equiv 0$, and only $[[\sigma_{11}]] = 0$ and $[[\sigma_{22}]] = 0$ are relevant.

Appendix B

Exact Solution to Eqs.(2.18) and (2.19)

First we introduct new variables

$$T = -\tau_1/(k/2R_p) \;, \quad \Gamma = \gamma_1/(k/\mu R_p), \quad X = x/R_p \tag{B.1a,b,c}$$

From Eq.(2.18) we then obtain

$$\Gamma = \frac{T}{M^2}\,[(1-M^2)F + \frac{1}{2}] \tag{B.2}$$

where

$$F = (1/T^2)dT/dX \tag{B.3}$$

Substitution of (B.2) into the dimensionless form of (2.19) yields by the use of (B.3)

$$2(1-M^2)\left(F\,\frac{dT}{dX} + T\,\frac{dF}{dX}\right) + 2\,\frac{dT}{dX} + \frac{1}{2}\,T^2 = 0 \tag{B.4}$$

Equation (B.4) can be rewritten as

$$2M\,\frac{dT}{T} = \left(\frac{1-M}{F + \frac{1}{2(1+M)}} - \frac{1+M}{F + \frac{1}{2(1-M)}}\right)dF \tag{B.5}$$

Integration of (B.5) gives

$$2M\ell nT = \ell n\left\{C\left|2F + \frac{1}{1+M}\right|^{1-M} \Big/ \left|2F + \frac{1}{1-M}\right|^{1+M}\right\} \tag{B.6}$$

where C is an integration constant.

At $X = x_p/R_p$ (the elastic-plastic boundary) we have

$$T = 1 \tag{B.7}$$

Since $\Gamma = -\frac{1}{2}$ at $x = x_p$, it follows from (B.2) that at $x = x_p$

$$F = F_p = -\ (1+M^2)/2(1-M^2) \tag{B.8}$$

By using (B.7) and (B.8) in (B.6) it then follows that

$$C = \frac{1-M}{1+M}\left(\frac{M^2}{1-M^2}\right)^M \tag{B.9}$$

Substitution of C in (B.6) gives

$$T = \left(\frac{1-M}{1+M}\right)^{1/2M} \frac{M}{(1-M^2)^{\frac{1}{2}}}\ \frac{[-2F-1/(1+M)]^{(1-M)/2M}}{[2F+1/(1-M)]^{(1+M)/2M}} \tag{B.10}$$

REFERENCES

(1) Achenbach, J.D. (1974). "Dynamic effects in brittle fracture". In Mechanics Today Vol. 1, ed. S. Nemat-Nasser, pp. 1-57. Oxford and New York: Pergamon Press.

(2) Freund, L.B. (1975). "Dynamic crack propagation". In The mechanics of fracture, ed. F. Erdogan, pp. 105-134. ASME AMD Vol. 19, New York: ASME.

(3) Kanninen, M.F. (1978). "A critical appraisal of solution techniques in dynamic fracture mechanics". In Numerical methods in fracture mechanics, eds. A.R. Luxmoore & D. R. J. Owen, pp. 612-633. Swansea: University College of Swansea Press.

(4) Rice, J.R. (1982). "Elastic-plastic crack growth". In: Mechanics of Solids, eds. H.G. Hopkins and M.J. Sewell, pp. 539-562. Oxford and New York: Pergamon Press.

(5) Achenbach, J.D. & Dunayevsky, V. (1984). "Crack growth under plane stress conditions in an elastic perfectly-plastic material". J. Mech. Phys. Solids, in press.

(6) Achenbach, J.D. and Li, Z.L. (1984). "Plane stress crackline fields for crack growth in an elastic perfectly-plastic material". J. Eng. Fract. Mech., in press.

(7) Slepyan, L.I. (1976). "Crack dynamics in an elastic-plastic body". Izv. Nauk SSSR, Mekhanika Tverdogo Tela, 12, No. 2, 144-153.

(8) Achenbach, J.D. & Dunayevsky, V. (1981). "Fields near a rapidly propagating crack tip in an elastic perfectly-plastic material". J. Mech. Phys. Solids, 29, p. 283.

(9) Achenbach, J.D. & Kanninen, M.F. (1978). "Crack tip plasticity in dynamic fracture mechanics". In Fracture mechanics, eds. N. Perrone et al., pp. 649-670. University of Virginia Press.

(10) Achenbach, J.D., Kanninen, M.F. & Popelar, C.H. (1981). "Crack tip fields for fast fracture of an elastic-plastic material". J. Mech. Phys. Solids, 29, 211-225.

(11) Dunayevsky, V. & Achenbach, J.D. (1982). "Boundary layer phenomenon in the plastic zone near a rapidiy propagating crack tip". Int. J. Solids Structures, 18, No. 1, pp. 1-12.

(12) Freund, L.B. & Douglas, A.S. (1982). "The influence of inertia on elastic-plastic antiplane-shear crack growth". J. Mech. Phys. Solids, 30, No. 1/2, 59-74.

(13) McClintock, F.A. & Irwin, G.R., (1965). "Plasticity aspects of fracture mechanics". In Fracture toughness testing and its applications, ASTM STP 381, Philadelphia: ASTM.

(14) Rice, J.R. (1968). "Mathematical analysis in the mechanics of fracture". In Fracture, ed. H. Liebowitz, pp. 191-311. New York and London: Academic Press.

(15) Creager, M. & Paris, P.C. (1967). "Elastic field equations for blunt cracks with reference to stress-corrosion cracking". Int. J. Fracture Mech., 3, 247-252.

(16) Drugan, W.J. & Rice, J.R. (1984), "Restrictions on quasi-statically moving surfaces of strong discontinuity". In Mechanics of material behavior, eds. G.J. Dvorak and R.T. Shield, Amsterdam: Elsevier Sci. Publ. Co.

(17) Hill, R.J. (1961), "Discontinuity relations in mechanics of solids". In Progress in solid mechanics, Vol. II, eds. Sneddon, I.N. & Hill, R., Amsterdam: North-Holland Publ. Co.

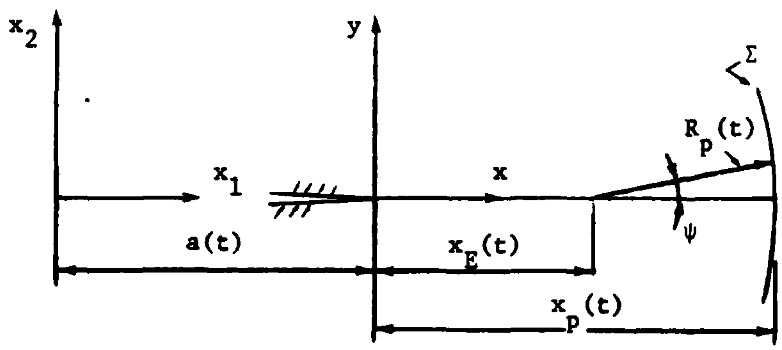

Fig. 1. Geometry for a propagating crack tip, with center of elastic field E, and elastic-plastic boundary Σ.

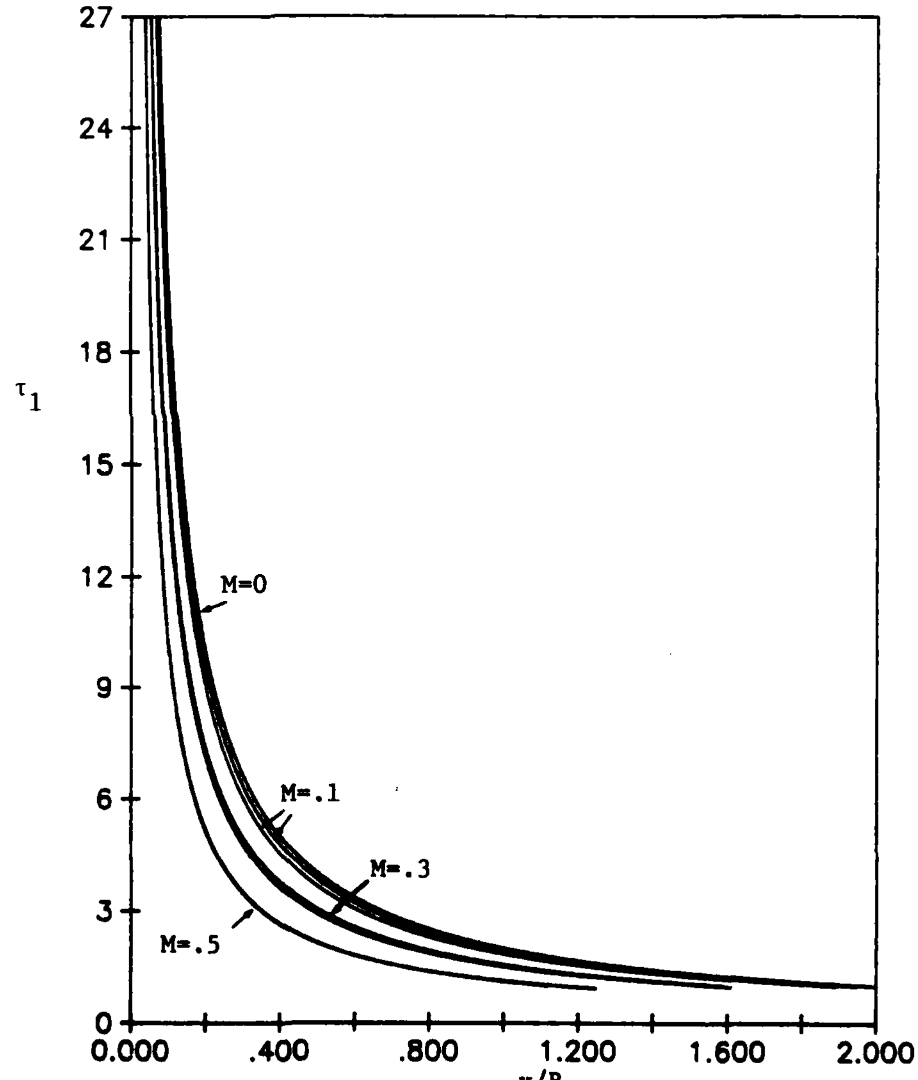

Fig. 2: Comparison of exact τ_1 and Eq.(2.40). Upper curves are exact curves.

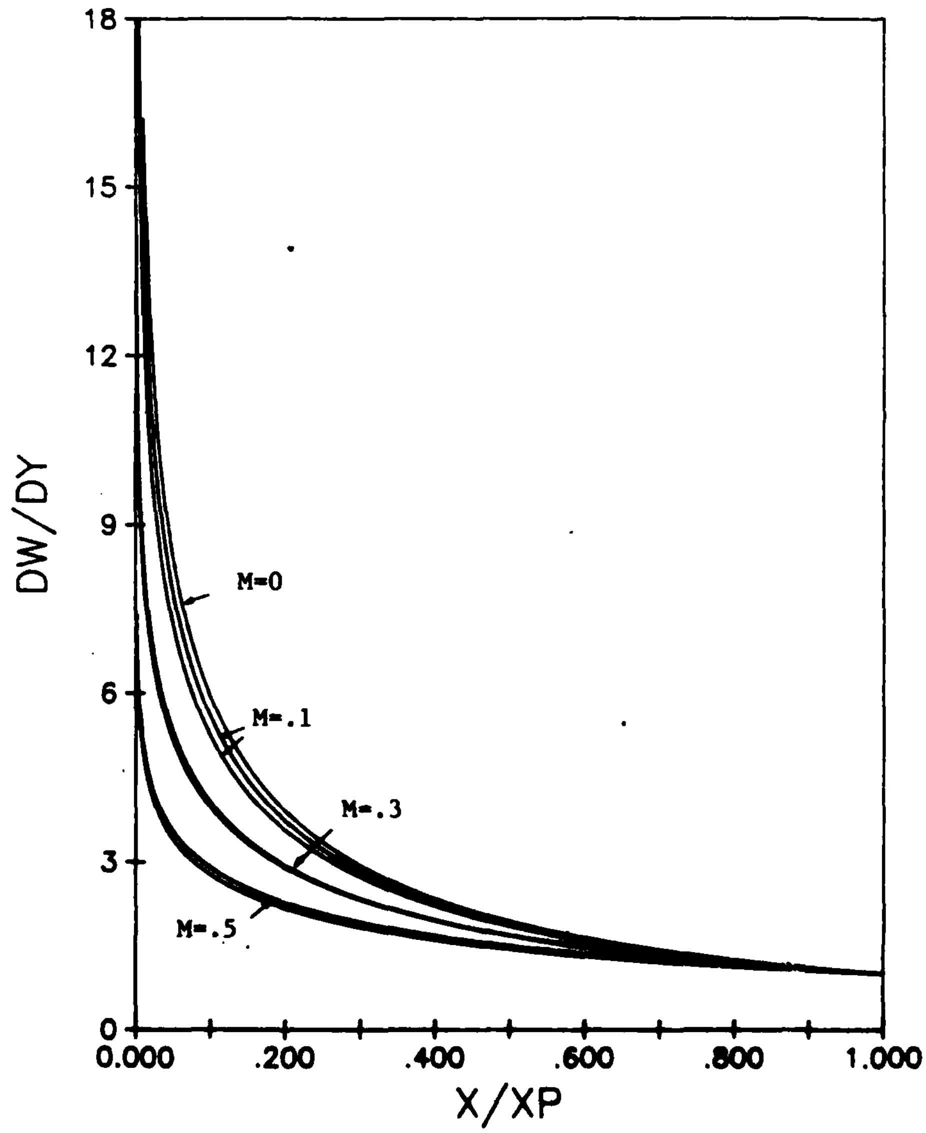

Fig. 3: Comparison of exact and approximate expressions for $\frac{\mu}{k}\gamma = 1 + \frac{\mu}{k}\gamma_y^p$. Lower curves are approximations according to Eq.(2.41).

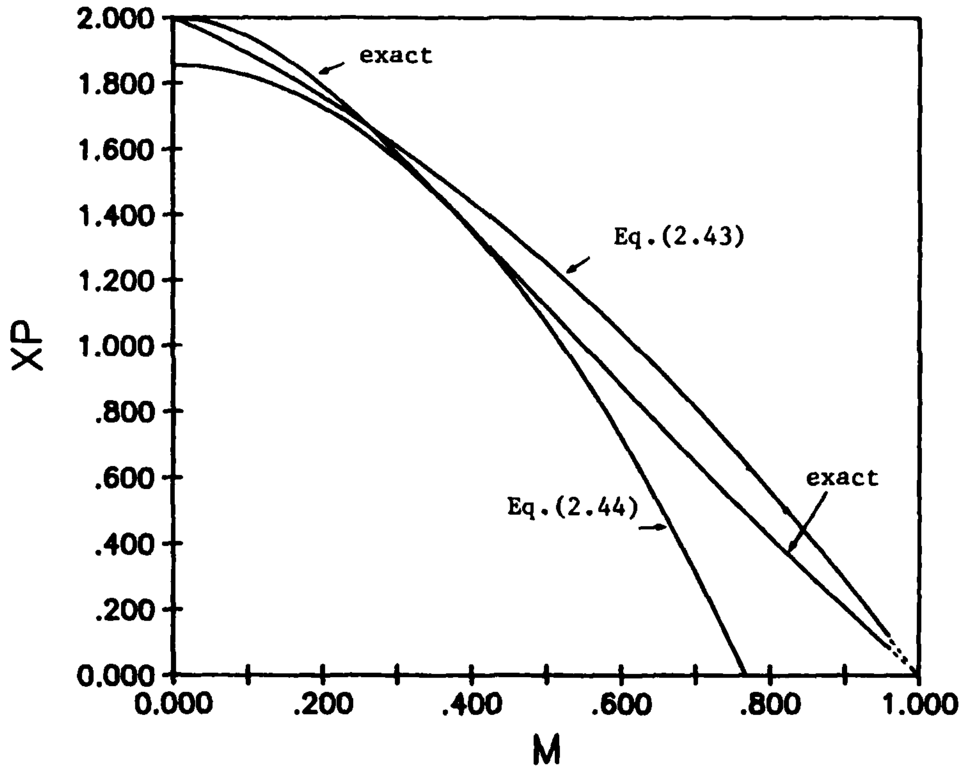

Fig. 4: Position of elastic-plastic boundary versus M.

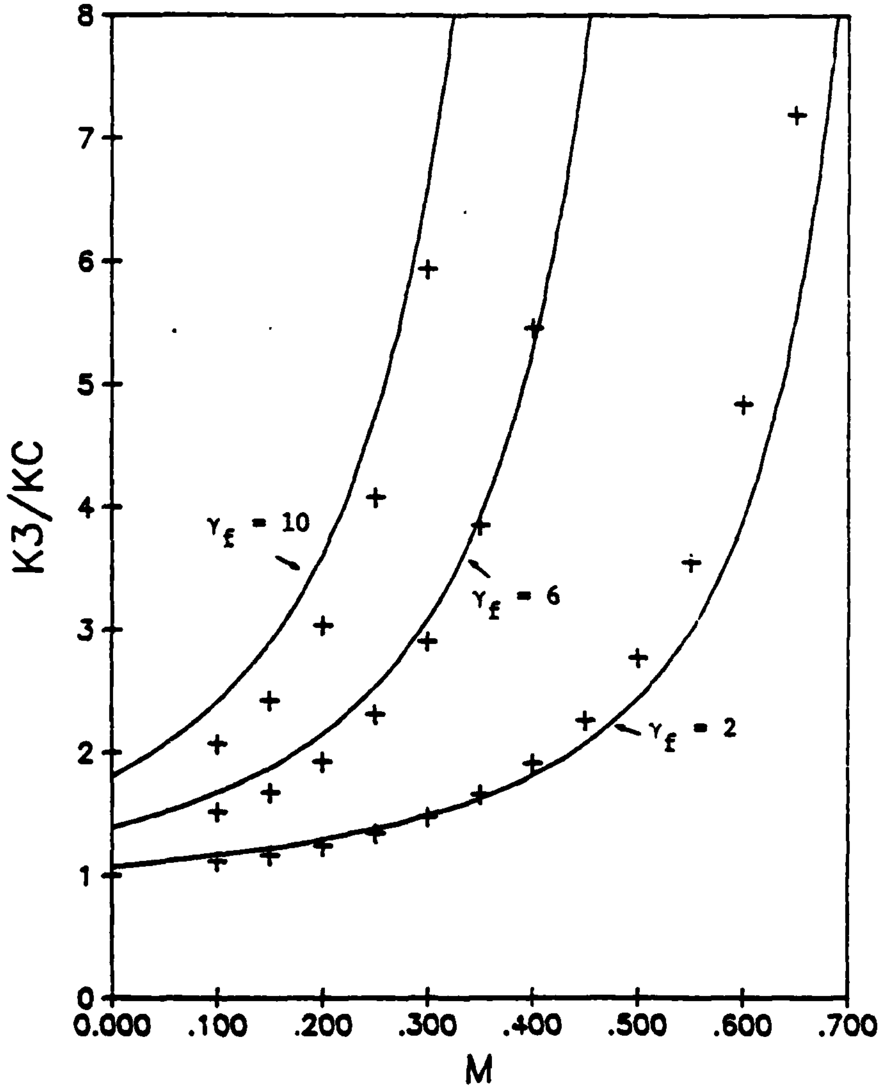

Fig. 5. Comparison between exact (+) and approximate (——) relation between K_{III}/K_c and M.

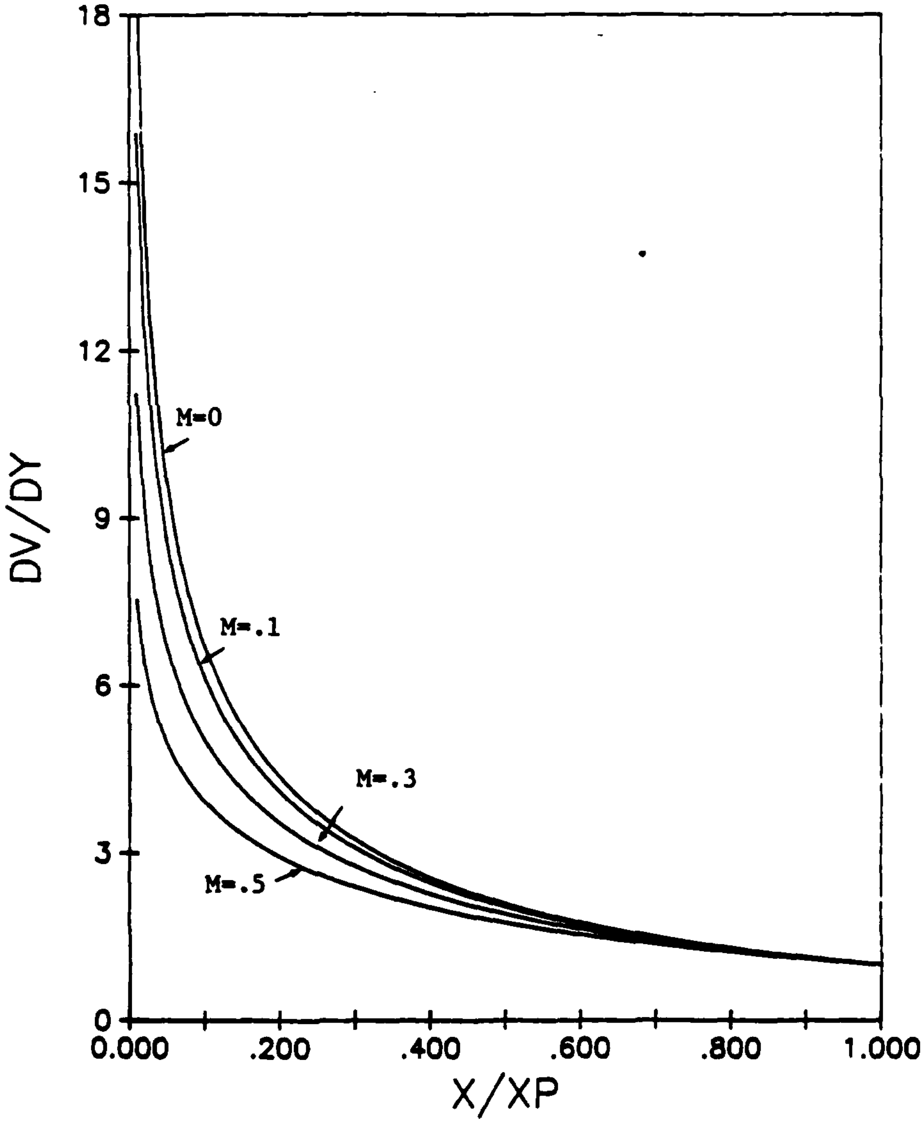

Fig. 6: Strain $\epsilon_y/(\epsilon_y)_{PB} = v_1/(v_1)_{PB}$ on crack line versus x/x_p, according to Eqs.(3.59)-(3.60).

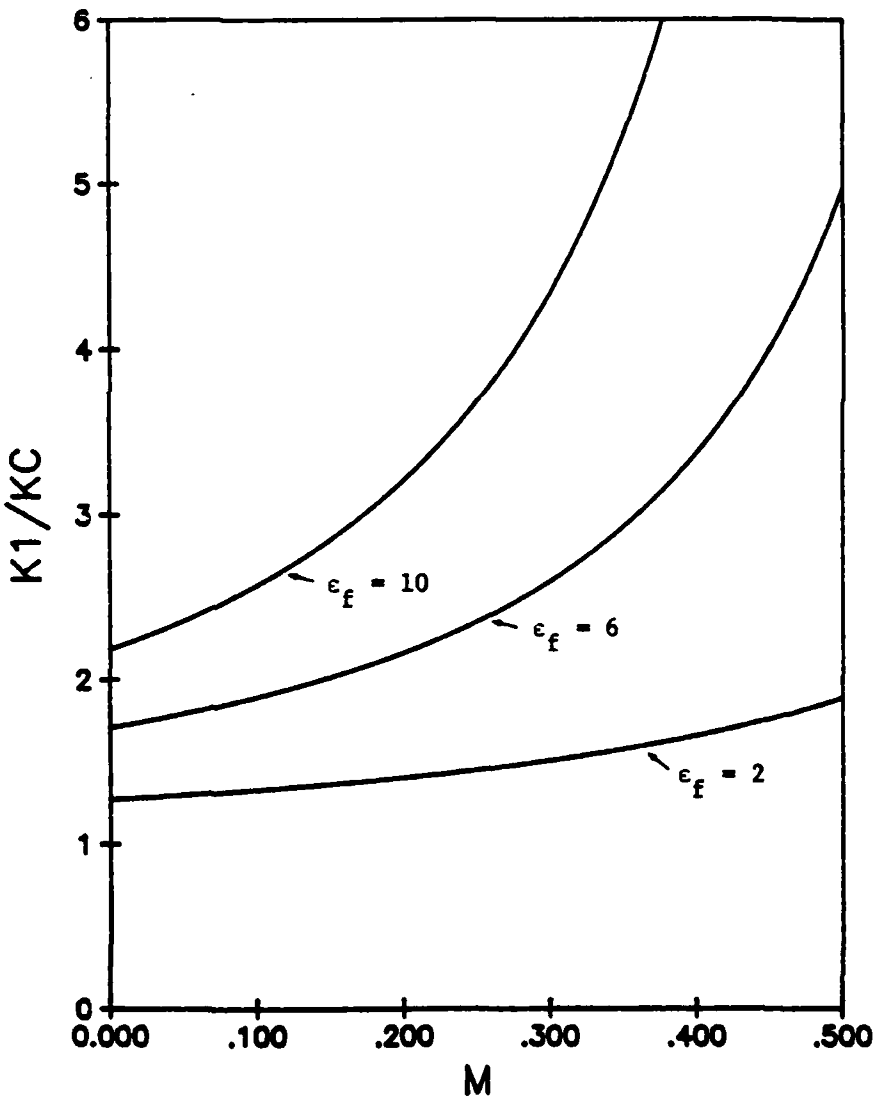

Fig. 7. K_I/K_{Ic} versus M for Mode-I plane stress.

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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20 ABSTRACT (Continue on reverse side if necessary and identify by block number)

For rapid crack propagation in an elastic perfectly-plastic material, explicit expressions have been obtained for the dynamic strains on the crack line, from the moving crack tip to the moving elastic-plastic boundary. The method of solution uses power series in the distance to the crack line, with coefficients which depend on the distance to the crack tip. Substitution of the expansions in the equations of motion, the yield condition (Huber-Mises) and the stress-strain relations, yields a system of nonlinear ordinary differential equations for the coefficients.

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These equations are exactly solvable for Mode-III, and they have been solved in an approximate manner for Mode-I plane stress. The crack-line fields have been matched to appropriate elastic fields at the elastic-plastic boundary. For both Mode-III and Mode-I plane stress, the plastic strains, which depend on the elastodynamic stress intensity factor and the crack-tip speed, have been used in conjunction with the crack growth criterion of critical plastic strain, to determine the relation between the far-field stress level and the crack-tip speed.

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